Weighted composition semigroups on spaces of holomorphic functions

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Semiflow

We call $\varphi := (\varphi_t)_{t\geq 0}$ a **semiflow** if $\varphi_t : \mathbb{D} \to \mathbb{D}$ is holomorphic and

1. $\varphi_0(z) = z$ for all $z \in \mathbb{D}$,
2. $\varphi_{t+s}(z) = (\varphi_t \circ \varphi_s)(z)$ for all $t, s \geq 0$ and $z \in \mathbb{D}$, and
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Examples:

- $\varphi_t(z) := \mathrm{e}^{-ct}z$ for $\mathrm{Re}(c) \geq 0$
- $\varphi_t(z) := \mathrm{e}^{-t}z + 1 - \mathrm{e}^{-t}$

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Semicocycle & co-semiflow

Let φ be a semiflow. We call $m := (m_t)_{t\geq 0}$ a **semicocycle** for φ if $m_t \colon \mathbb{D} \to \mathbb{C}$ is holomorphic and

1. $m_0(z) = 1$ for all $z \in \mathbb{D}$,
2. $m_{t+s}(z) = m_t(z)m_s(\varphi_t(z))$ for all $t, s \geq 0$ and $z \in \mathbb{D}$, and
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The tuple (m, φ) is called a **co-semiflow**.

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Examples:

- $m_t(z) := \varphi_t'(z)$
- $m_t(z) := \exp(\int_0^t g(\varphi_s(z))\mathrm{d}s)$ for $g \in \mathcal{H}(\mathbb{D})$

Weighted composition semigroup

Let

- $(\mathcal{F}(\mathbb{D}), \|\cdot\|)$ be a Banach space of holomorphic functions on $\mathbb{D}$,
- (m, φ) a co-semiflow,
- $C_{m,\varphi}(t)f := m_t \cdot (f \circ \varphi_t) \in \mathcal{F}(\mathbb{D})$ for all $t \geq 0$ and $f \in \mathcal{F}(\mathbb{D})$.

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$\|\cdot\|$-strongly continuous:

- $C_{m,\varphi}(t) \in \mathcal{L}(\mathcal{F}(\mathbb{D}))$ for all $t \geq 0$,
- $[0, \infty) \to (\mathcal{F}(\mathbb{D}), \|\cdot\|)$, $t \mapsto C_{m,\varphi}(t)f$, continuous for all $f \in \mathcal{F}(\mathbb{D})$.

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Theorem (Siskakis 1986, König 1990, Wu 2021)

Let

- $p \in [1, \infty)$,
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Theorem (Gallardo-Gutiérrez, Siskakis, Yakubovich 2021)

Let

- $(\mathcal{F}(\mathbb{D}), \|\cdot\|)$ be a Banach space of holomorphic functions on $\mathbb{D}$,
- $H^\infty \subseteq \mathcal{F}(\mathbb{D}) \subseteq \mathcal{B}_1$,
- φ a non-trivial semiflow,
- m a semicocycle for φ s.t. $(C_{m,\varphi}(t))_{t\geq 0}$ is a semigroup on $\mathcal{F}(\mathbb{D})$.

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- Observation: There are non-trivial weighted composition semigroups on such spaces $\mathcal{F}(\mathbb{D})$.
- Example: $\varphi_t(z) := \mathrm{e}^{-ct}z$ for $\mathrm{Re}(c) > 0$, $m_t := \varphi_t'$ and $\mathcal{F}(\mathbb{D}) := H^\infty$

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Question:

Is there a weaker concept of strong continuity for weighted composition semigroups on such spaces?

Saks space & mixed topology

Saks space (Wiweger 1961, Cooper 1978)

Let

- $(X, \|\cdot\|)$ be Banach and τ a coarser Hausdorff l.c. topology on X,
- there exist a norming system of continuous seminorms Γ_τ of τ.

Then the triple $(X, \|\cdot\|, \tau)$ is called a **Saks space**.

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- **Example:** Let $X := H^\infty$, $\|\cdot\| := \|\cdot\|_\infty$, $\tau := \tau_{\mathrm{co}}$. Then

$$\|f\|_\infty = \sup_{\substack{K\subset\mathbb{D}\\ \text{compact}}} \sup_{z\in K} |f(z)|, \quad f \in H^\infty.$$

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$\Rightarrow (H^\infty, \|\cdot\|_\infty, \tau_{\mathrm{co}})$ is a Saks space.

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Mixed topology (Wiweger 1961)

Let $(X, \|\cdot\|, \tau)$ be a Saks space.

- **Mixed topology** $\gamma := \gamma(\|\cdot\|, \tau)$:⇔ the finest linear topology s.t. $\gamma = \tau$ on $\|\cdot\|$-bounded sets.
- $(X, \|\cdot\|, \tau)$ (seq.) complete :⇔ (X, γ) (seq.) complete.

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- **Example:** γ of $(H^\infty, \|\cdot\|_\infty, \tau_{\mathrm{co}})$ is generated by the seminorms

$$|f|_w := \sup_{z\in\mathbb{D}} |f(z)|w(z), \quad f \in H^\infty,\ w \in \mathcal{C}_0^+(\mathbb{D}).$$

Semigroups on Saks spaces

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Let $(X, \|\cdot\|, \tau)$ be a Saks space. A semigroup $(T(t))_{t\geq 0}$ on X is called

- **locally bounded** if for all $t_0 \geq 0$: $\sup_{t\in[0,t_0]} \|T(t)\|_{\mathcal{L}(X)} < \infty$;
- **γ-strongly continuous** if
  1. for all $t \geq 0$: $T(t) \in \mathcal{L}(X, \gamma)$,
  2. for all $x \in X$ the map $[0, \infty) \to (X, \gamma)$, $t \mapsto T(t)x$, is continuous;
- **locally γ-equicontinuous** if for all $t_0 \geq 0$ with $I := [0, t_0]$ it holds

$$\forall\, p \in \Gamma_\gamma\, \exists\, \widetilde{p} \in \Gamma_\gamma, C \geq 0\, \forall\, t \in I, x \in X: \; p(T(t)x) \leq C\widetilde{p}(x).$$

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- γ-strongly continuous & locally γ-equicontinuous $\Rightarrow$ locally bounded.

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- γ-strongly continuous & locally γ-equicontinuous $\Rightarrow$ locally bounded.
- Kühnemund 2001, Farkas 2003, Federico, Rosestolato 2020: Let $(X, \|\cdot\|, \tau)$ be a sequentially complete Saks space. Then: τ-bi-continuous $\Leftrightarrow$ 2 & locally sequentially γ-equicontinuous.

Theorem (K 2022)

Let

- $(\mathcal{F}(\mathbb{D}), \|\cdot\|, \tau_{\mathrm{co}})$ be a Saks space of holomorphic functions on $\mathbb{D}$,
- (m, φ) a co-semiflow,
- $(C_{m,\varphi}(t))_{t\geq 0}$ a locally bounded weighted composition semigroup on $\mathcal{F}(\mathbb{D})$ w.r.t. (m, φ).

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Then the following assertions hold.

1. $(C_{m,\varphi}(t))_{t\geq 0}$ is γ-strongly continuous and locally γ-equicontinuous.
2. Let $(\mathcal{F}(\mathbb{D}), \|\cdot\|, \tau_{\mathrm{co}})$ be sequentially complete. Then $(C_{m,\varphi}(t))_{t\geq 0}$ is τ_{co}-bi-continuous, and if $(\mathcal{F}(\mathbb{D}), \|\cdot\|)$ is also reflexive, $\|\cdot\|$-str. cont.

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Corollary (K 2022)

Let

- (m, φ) a be co-semiflow,
- $m_t \in H^\infty$ for all $t \geq 0$.

Then $(C_{m,\varphi}(t))_{t\geq 0}$ is γ-strongly continuous, locally γ-equicontinuous and τ_{co}-bi-continuous on H^∞.

$\|\cdot\|$-Generator

Let

- $(X, \|\cdot\|)$ be a Banach space,
- $(T(t))_{t\geq 0}$ a $\|\cdot\|$-strongly continuous semigroup on X.

Then the generator $(A_{\|\cdot\|}, D(A_{\|\cdot\|}))$ of $(T(t))_{t\geq 0}$ is given by

$$D(A_{\|\cdot\|}) := \left\{ x \in X \mid \|\cdot\|\text{-}\lim_{t\to 0+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}$$

and

$$A_{\|\cdot\|}x := \|\cdot\|\text{-}\lim_{t\to 0+} \frac{T(t)x - x}{t}, \quad x \in D(A_{\|\cdot\|}).$$

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Theorem (König 1990)

Let

- $p \in [1, \infty)$,
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Then the $\|\cdot\|_p$-generator of $(A_{\|\cdot\|_p}, D(A_{\|\cdot\|_p}))$ of $(C_{m,\varphi}(t))_{t\geq 0}$ fulfils

$$D(A_{\|\cdot\|_p}) = \{ f \in H^p \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in H^p \}, \quad A_{\|\cdot\|_p} f = \dot{\varphi}_0 f' + \dot{m}_0 f, \ f \in D(A_{\|\cdot\|_p}).$$

γ-Generator

Let

- $(X, \|\cdot\|, \tau)$ be a Saks space,
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Then the γ-generator $(A_\gamma, D(A_\gamma))$ of $(T(t))_{t\geq 0}$ is given by

$$D(A_\gamma) := \left\{ x \in X \mid \gamma\text{-}\lim_{t\to 0+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}$$

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- $(C_{m,\varphi}(t))_{t\geq 0}$ a locally bounded weighted composition semigroup on $\mathcal{F}(\mathbb{D})$ w.r.t. (m, φ).

Then the following assertions hold.

1. The γ-generator of $(A_\gamma, D(A_\gamma))$ of $(C_{m,\varphi}(t))_{t\geq 0}$ fulfils

$$D(A_\gamma) = \{f \in \mathcal{F}(\mathbb{D}) \,|\, \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\mathbb{D})\}, \; A_\gamma f = \dot{\varphi}_0 f' + \dot{m}_0 f, \; f \in D(A_\gamma).$$

2. If $(C_{m,\varphi}(t))_{t\geq 0}$ is $\|\cdot\|$-strongly continuous, then $D(A_\gamma) = D(A_{\|\cdot\|})$ and $A_\gamma = A_{\|\cdot\|}$.

Theorem (K 2022)

Let

- $(\mathcal{F}(\mathbb{D}), \|\cdot\|, \tau_{\mathrm{co}})$ be a sequentially complete Saks space of holomorphic functions on $\mathbb{D}$,
- (m, φ) a co-semiflow,
- $(C_{m,\varphi}(t))_{t\geq 0}$ a locally bounded weighted composition semigroup on $\mathcal{F}(\mathbb{D})$ w.r.t. (m, φ).

Then the following assertions hold.

1. The γ-generator of $(A_\gamma, D(A_\gamma))$ of $(C_{m,\varphi}(t))_{t\geq 0}$ fulfils
$$D(A_\gamma) = \{f \in \mathcal{F}(\mathbb{D}) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\mathbb{D})\},\ A_\gamma f = \dot{\varphi}_0 f' + \dot{m}_0 f,\ f \in D(A_\gamma).$$
2. If $(C_{m,\varphi}(t))_{t\geq 0}$ is $\|\cdot\|$-strongly continuous, then $D(A_\gamma) = D(A_{\|\cdot\|})$ and $A_\gamma = A_{\|\cdot\|}$.
3. Let $\big[(m, \varphi), \mathcal{F}(\mathbb{D})\big]$ be the space of $\|\cdot\|$-strong continuity of $(C_{m,\varphi}(t))_{t\geq 0}$. Then
$$\big[(m, \varphi), \mathcal{F}(\mathbb{D})\big] = \overline{\{f \in \mathcal{F}(\mathbb{D}) \mid \dot{\varphi}_0 f' + \dot{m}_0 f \in \mathcal{F}(\mathbb{D})\}}^{\|\cdot\|}.$$