On the Lumer–Phillips theorem for bi-continuous semigroups

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joint work with

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Theorem (Lumer, Phillips 1961)

Let

- $(X, \|\cdot\|)$ be a Banach space,
- $(A, D(A))$ a $\|\cdot\|$-densely defined, $\|\cdot\|$-dissipative operator,
- $\mathrm{Ran}(\lambda - A)$ $\|\cdot\|$-dense in X for some $\lambda > 0$.

Then the $\|\cdot\|$-closure $(\overline{A}, D(\overline{A}))$ generates a $\|\cdot\|$-strongly continuous contraction semigroup $(T(t))_{t\geq 0}$ on X.

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Example

$$D(\Delta) := \{f \in \mathrm{C_b}(\mathbb{R}^d) \,|\, \forall p \geq 1 : \; f \in \mathrm{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d), \; \Delta f \in \mathrm{C_b}(\mathbb{R}^d)\}, \; d \geq 2$$

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Lorenzi, Bertoldi 2007: $(\Delta, D(\Delta))$ is the generator of the Gauß–Weierstraß sg $(T(t))_{t\geq 0}$ on $\mathrm{C_b}(\mathbb{R}^d)$ given by $T(0)f := f$ and

$$T(t)f(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} f(y) e^{\frac{-|y-x|^2}{4t}} \mathrm{d}y, \quad x \in \mathbb{R}^d, \; f \in \mathrm{C_b}(\mathbb{R}^d), \; t > 0.$$

Saks space & mixed topology

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Saks space (Wiweger 1961, Cooper 1978)

Let

- $(X, \|\cdot\|)$ be Banach and τ a coarser Hausdorff l.c. topology on X,
- there exist a norming system of continuous seminorms Γ_τ of τ.

Then the triple $(X, \|\cdot\|, \tau)$ is called a **Saks space.**

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Mixed topology (Wiweger 1961)

Let $(X, \|\cdot\|, \tau)$ be a Saks space.

- **Mixed topology** $\gamma := \gamma(\|\cdot\|, \tau) :\Leftrightarrow$ the finest linear topology s.t. $\gamma = \tau$ on $\|\cdot\|$-bounded sets.
- $(X, \|\cdot\|, \tau)$ (seq.) complete $:\Leftrightarrow$ (X, γ) (seq.) complete.

Bi-continuous semigroup

Bi-continuous semigroup (Kühnemund 2001)

Let $(X, \|\cdot\|, \tau)$ be a sequentially complete Saks space. An sg $(T(t))_{t\geq 0}$ in $\mathcal{L}(X)$ is called τ-**bi-continuous** if

1. $(T(t))_{t\geq 0}$ is τ-strongly continuous,
2. $\exists\, M \geq 1, \omega \in \mathbb{R}\ \forall t \geq 0$: $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$,
3. $\forall\, (x_n)_{n\in\mathbb{N}}$ in X, $x \in X$ with $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$ and $\tau\text{-}\lim_{n\to\infty} x_n = x$:

$$\tau\text{-}\lim_{n\to\infty} T(t)(x_n - x) = 0$$

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- Farkas 2003, Federico, Rosestolato 2020: τ-bi-continuous $\Leftrightarrow$ γ-strongly continuous & locally sequentially γ-equicontinuous.

Equicontinuity, equitightness & submixed topology

Equicontinuity & equitightness (Farkas 2003)

An sg $(T(t))_{t\geq 0}$ on a Saks space $(X, \|\cdot\|, \tau)$ is called

- γ-**equicontinuous** if

$$\forall\, p \in \Gamma_\gamma\, \exists\, \widetilde{p} \in \Gamma_\gamma, C \geq 0\, \forall\, t \geq 0, x \in X : \; p(T(t)x) \leq C\widetilde{p}(x).$$

- **equitight** if

$$\forall\, \varepsilon > 0, p \in \Gamma_\tau\, \exists\, \widetilde{p} \in \Gamma_\tau, C \geq 0\, \forall\, t \geq 0, x \in X : \; p(T(t)x) \leq C\widetilde{p}(x) + \varepsilon\|x\|.$$

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Submixed topology (Wiweger 1961, Cooper 1978)

Let $(X, \|\cdot\|, \tau)$ be a Saks space and Γ_τ a norming system of continuous seminorms of τ. For $(p_n)_{n\in\mathbb{N}} \subseteq \Gamma_\tau$ and $(a_n)_{n\in\mathbb{N}} \in c_0^+$ set

$$|||x|||_{(p_n, a_n)_{n\in\mathbb{N}}} := \sup_{n\in\mathbb{N}} p_n(x) a_n, \quad x \in X.$$

Submixed topology $\gamma_s := \gamma_s(\|\cdot\|, \tau) :\Leftrightarrow$ Hausdorff locally convex topology generated by all such seminorms.

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Dissipativity (Albanese, Jornet 2016)

Let

- (X, υ) be a Hausdorff locally convex space,
- Γ_υ a system of continuous seminorms of υ.

A linear operator $(A, D(A))$ on X is called Γ_υ**-dissipative** if

$$\forall\, \lambda > 0,\, x \in D(A),\, p \in \Gamma_\upsilon : \; p((\lambda - A)x) \geq \lambda p(x).$$

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Bi-dissipativity (Budde, Wegner 2022)

Let

- $(X, \|\cdot\|, \tau)$ be a sequentially complete Saks space,
- Γ_τ norming system of continuous seminorms of τ.

Then $(A, D(A))$ **bi-dissipative** $:\Leftrightarrow$ $(A, D(A))$ Γ_τ-dissipative.

Theorem (K, Seifert 2022)

Let

- $(X, \|\cdot\|, \tau)$ be a complete Saks space,
- $(A, D(A))$ a γ-densely defined, Γ_γ-dissipative operator,
- $\mathrm{Ran}(\lambda - A)$ γ-dense in X for some $\lambda > 0$.

Then the following assertions hold:

1. The γ-closure $(\overline{A}, D(\overline{A}))$ generates a γ-strongly continuous, γ-equicontinuous semigroup $(T(t))_{t\geq 0}$ on X.
2. If Γ_γ is norming, then $(T(t))_{t\geq 0}$ is a contraction semigroup.
3. If Γ_γ is norming and $\gamma = \gamma_s$, then $(T(t))_{t\geq 0}$ is equitight.

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Theorem (Budde, Wegner 2022)

- Let $(X, \|\cdot\|, \tau)$ be a seq. comp. Saks space s.t. (X, γ_s) is complete,
- $(A, D(A))$ a sequentially γ-densely defined, bi-dissipative operator,
- $\mathrm{Ran}(\lambda - A)$ sequentially γ-dense in X for some $\lambda > 0$.

Then γ_s-closure $(\overline{A}, D(\overline{A}))$ generates a τ-bi-continuous contraction sg.

Proposition (K, Seifert 2022)

Let

- $(X, \|\cdot\|, \tau)$ be a complete, C-sequential Saks space,
- $(A, D(A))$ the generator of a τ-bi-continuous semigroup $(T(t))_{t\geq 0}$ on X.

Then the following assertions are equivalent:

1. $(T(t))_{t\geq 0}$ is γ-equicontinuous.
2. There is a system of continuous seminorms Γ_γ of the mixed topology γ such that $(A, D(A))$ is Γ_γ-dissipative.

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- Kraaij 2016, K, Schwenninger 2022: If $(X, \|\cdot\|, \tau)$ is a sequentially complete, C-sequential Saks space, then any τ-bi-continuous sg $(T(t))_{t\geq 0}$ on X is quasi-γ-equicontinuous.

Corollary (K, Seifert 2022)

Let

- $(X, \|\cdot\|, \tau)$ be a complete Saks space,
- $(A, D(A))$ a Γ_γ-dissipative operator,
- its γ-dual operator $(A', D(A'))$ a $\|\cdot\|_{X'_\gamma}$-dissipative.

Then the following assertions hold:

1. The γ-closure $(\overline{A}, D(\overline{A}))$ generates a γ-strongly continuous, γ-equicontinuous semigroup $(T(t))_{t\geq 0}$ on X.
2. If Γ_γ is norming, then $(T(t))_{t\geq 0}$ is a contraction semigroup.
3. If Γ_γ is norming and $\gamma = \gamma_s$, then $(T(t))_{t\geq 0}$ is equitight.

Corollary (K, Seifert 2022)

Let

- $(X, \|\cdot\|, \tau)$ be a complete, semi-reflexive Saks space,
- $(A, D(A))$ a γ-densely defined, Γ_γ-dissipative operator,
- $\mathrm{Ran}(\lambda - A) = X$ for some $\lambda > 0$.

Then the following assertions hold:

1. $(A, D(A))$ generates a γ-equicontinuous, γ-strongly continuous semigroup $(T(t))_{t\geq 0}$ on X.
2. If Γ_γ is norming, then $(T(t))_{t\geq 0}$ is a contraction semigroup.
3. If Γ_γ is norming and $\gamma = \gamma_s$, then $(T(t))_{t\geq 0}$ is equitight.